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# Generic properties of 2-step nilpotent Lie algebras and torsion-free groups

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To define the notion of a generic property of finite dimensional 2-step nilpotent Lie algebras we use standard correspondence between such Lie algebras and points of an appropriate algebraic variety, where a negligible set is one contained in a proper Zariski-closed subset. We compute the maximal dimension of an abelian subalgebra of a generic Lie algebra and give a sufficient condition for a generic Lie algebra to admit no surjective homomorphism onto a non-abelian Lie algebra of a given dimension. Also we consider analogous questions for finitely generated torsion free nilpotent groups of class 2.

## 1 Introduction and Results

### 1.1 Nilpotent Lie algebras of class 2

Consider a finite-dimensional 2-step nilpotent Lie algebra  $L$ . Denote by  $S$  the commutator subalgebra of  $L$ . Let  $z_1, \dots, z_t$  be a basis of  $S$ , and let  $V \subset L$  be such a subspace that  $L = V \oplus S$ . Then the product of two elements  $x = \bar{x} + \bar{\bar{x}}$  and  $y = \bar{y} + \bar{\bar{y}}$  of  $L$  with  $\bar{x}, \bar{y} \in V$  and  $\bar{\bar{x}}, \bar{\bar{y}} \in S$  has the form

$$[x, y] = \varphi_1(\bar{x}, \bar{y})z_1 + \dots + \varphi_t(\bar{x}, \bar{y})z_t \quad (1.1)$$

for some  $t$ -tuple of alternating bilinear forms  $\Phi = \Phi(L) = (\varphi_1, \dots, \varphi_t)$  on  $V$ .

On the other hand, given vector spaces  $S$  and  $V$ , a basis  $z_1, \dots, z_t$  of  $S$  and a  $t$ -tuple  $\Phi = (\varphi_1, \dots, \varphi_t)$  of alternating bilinear forms on  $V$ , one can define the product of two elements of  $L = L(\Phi) = V \oplus S$  by (1.1). Obviously,  $L(\Phi)$  is a 2-step nilpotent Lie algebra and  $S$  is a central subalgebra of  $L(\Phi)$ .

Let  $z'_1, \dots, z'_t$  be a new basis of  $S$  and let  $C = (c_{ij})$  be the transformation matrix from old to new coordinates. Then (1.1) can be written in the following form:

$$[x, y] = \sum_{i=1}^t \varphi_i(\bar{x}, \bar{y})z_i = \sum_{i=1}^t \varphi_i(\bar{x}, \bar{y}) \sum_{j=1}^t c_{ji}z'_j = \sum_{j=1}^t \left( \sum_{i=1}^t \varphi_i(\bar{x}, \bar{y})c_{ji} \right) z'_j = \sum_{i=1}^t \varphi'_i(\bar{x}, \bar{y})z'_i, \quad (1.2)$$

where

$$\varphi'_i = \sum_{j=1}^t c_{ij}\varphi_j, \quad (1.3)$$

That is  $C^\perp$  is the transformation matrix to go from  $\Phi$  to  $\Phi' = (\varphi'_1, \dots, \varphi'_t)$ . This gives the following properties of a  $t$ -tuple  $\Phi$ .

**Proposition 1.** *The alternating bilinear forms of the  $t$ -tuple  $\Phi(L)$  corresponding to a Lie algebra  $L$  are linearly independent. And conversely, if the forms of a  $t$ -tuple  $\Phi$  are linearly independent and  $L(\Phi) = V \oplus S$  is a corresponding Lie algebra, then  $S = [L(\Phi), L(\Phi)]$ .*

*Proof.* Suppose, on the contrary, that for some Lie algebra  $L$  the forms of  $\Phi(L)$  are linearly dependent. Then there exists non-trivial linear combination  $\alpha_1\varphi_1 + \dots + \alpha_t\varphi_t = 0$ . It follows from (1.3) that, putting  $c_{tj} = \alpha_j$ , one can choose a basis  $z'_1, \dots, z'_t$  of  $S$  in such a way that  $\varphi'_t \equiv 0$ . This contradicts with the fact that  $z'_t$  belongs to the commutator subalgebra  $S$ .

Conversely, if the commutator subalgebra  $[L(\Phi), L(\Phi)]$  is strictly less than  $S$ , then there exists a basis  $z'_1, \dots, z'_t$  of  $S$  such that  $z_1, \dots, z_k$ , where  $k < t$ , generate  $[L(\Phi), L(\Phi)]$ . Then the corresponding bilinear form  $\varphi'_t$ , which is a non-trivial linear combination of the forms of  $\Phi$ , identically equals zero.  $\square$

**Proposition 2.** *If the linear spans of  $t$ -tuples  $\Phi$  and  $\Phi'$  coincide, then the corresponding Lie algebras  $L(\Phi)$  and  $L(\Phi')$  are isomorphic.*

From now on we will assume that a ground field is infinite. It is appropriate to introduce some notation that will be used throughout the paper. We denote by  $\mathfrak{N}_2^K(n, t)$  (or simply by  $\mathfrak{N}_2(n, t)$ , if it is not misleading) the set of all 2-step nilpotent Lie algebras  $L$  over a field  $K$  such that  $\dim L/[L, L] = n$  and  $\dim[L, L] = t$ . Let  $B_n(K)$  be the set of all alternating bilinear forms on an  $n$ -dimensional space  $V$  over  $K$  and  $\mathbb{P}(B_n^t)$  be the projective space corresponding to the  $t$ -th direct power of  $B_n(K)$ . Put  $M_{n,t} = \{(\varphi_1 : \dots : \varphi_t) \in \mathbb{P}(B_n^t) \mid \varphi_1, \dots, \varphi_t \text{ are linearly independent}\}$ .  $M_{n,t}$  is a Zariski-open subset of  $\mathbb{P}(B_n^t)$ .

**Definition 1.** We say that  $\mathcal{A}$  is a generic property of  $\mathfrak{N}_2(n, t)$  (a generic Lie algebra of  $\mathfrak{N}_2(n, t)$  has a property  $\mathcal{A}$ ) if the set of points of  $M_{n,t}$  corresponding to the algebras without the property  $\mathcal{A}$  is contained in some proper Zariski-closed subset.

*Remark.* Since  $M_{n,t}$  is irreducible, the dimension of any its closed subset is strictly less than the dimension of  $M_{n,t}$ . (see. [7])

The following theorem, which holds true for any infinite field, gives a simple example of a generic property.

**Theorem 1.** *For a generic Lie algebra  $L \in \mathfrak{N}_2(n, t)$  we have  $\dim Z(L) = 2$  if  $t = 1$  and  $n$  is odd, and  $Z(L) = [L, L]$  otherwise.*

Here  $Z(L)$  is the center of a Lie algebra  $L$ .

The next theorem allows to compute the maximal dimension of an abelian subalgebra of a generic Lie algebra.

**Theorem 2.** *If a ground field  $K$  is algebraically closed and  $t > 1$ , then any Lie algebra of  $\mathfrak{N}_2^K(n, t)$  contains a commutative subalgebra of dimension  $s = \lfloor \frac{2n+t^2+3t}{t+2} \rfloor$ . For any infinite ground field  $K$  a generic Lie algebra of  $\mathfrak{N}_2^K(n, t)$  doesn't have any commutative subalgebra of dimension  $s + 1$ .*

This theorem immediately follows from [6]. However for the convenience of the reader we will give some details in Section 3.2. Also in this section we adduce an example of a Lie algebra from  $\mathfrak{N}_2^{\mathbb{R}}(4, 3)$  without commutative subalgebras of dimension 5. Which shows that the condition for the ground field being algebraically closed can not be ignored.

*Remark.* The structure of Heisenberg Lie algebras, that is 2-step nilpotent Lie algebras with one-dimensional center, is well known. The dimension  $m$  of such an algebra is always odd, and the dimension of a maximal commutative subalgebra is  $\frac{m+1}{2}$ .

The main result of the present paper is the following theorem, which holds true for an arbitrary infinite ground field.

**Definition 2.** We say that a nilpotent Lie algebra  $L$  of class 2 has a property  $\mathcal{S}(n_0, t_0)$  ( $1 \leq t_0 \leq \frac{n_0(n_0-1)}{2}$ ), if  $L$  admits a surjective homomorphism onto a Lie algebra of  $\mathfrak{N}_2(n_0, t_0)$ .

**Theorem 3.** If the positive integers  $n$ ,  $n_0$ ,  $t$  and  $t_0$  satisfy the inequality

$$t < \frac{n(n-1)}{2} - \frac{nn_0}{t_0} + \frac{n_0^2}{t_0} + t_0 - \frac{n_0(n_0-1)}{2}, \quad (1.4)$$

then a generic Lie algebra of  $\mathfrak{N}_2(n, t)$  does not have the property  $\mathcal{S}(n_0, t_0)$ . If  $n$ ,  $n_0$ ,  $t$  and  $t_0$ , where  $n \geq n_0$ , satisfy the inequality

$$t \geq \frac{n(n-1)}{2} - \frac{n_0(n_0-1)}{2} + t_0, \quad (1.5)$$

then the property  $\mathcal{S}(n_0, t_0)$  is true on  $\mathfrak{N}_2(n, t)$ .

The following statement is an immediate corollary of Theorem 3.

**Corollary.** A generic Lie algebra of  $\mathfrak{N}_2(n, t)$  does not admit a surjective homomorphism onto a non-commutative Lie algebra of dimension  $N < n$ , if

$$t < \frac{n^2}{2} - n \left( N - \frac{1}{2} \right) + \frac{N(N-1)}{2} + 1. \quad (1.6)$$

*Remark.* Obviously, if  $n < N \leq n+t$ , then any Lie algebra  $L \in \mathfrak{N}_2(n, t)$  admits a surjective homomorphism onto a non-commutative Lie algebra of dimension  $N$ . It is enough to take a quotient group of  $L$  by any central subalgebra of dimension  $n+t-N < t$ .

Actually, if a ground field is algebraically closed, then the set of point of  $M_{n,t}$  corresponding to the algebras with property  $\mathcal{S}(n_0, t_0)$  forms a closed subset for any values of parameters (see Lemmas 2 and ??). It means that there are only two possibilities: either this subset coincides with  $M_{n,t}$  and all the algebras have the property  $\mathcal{S}(n_0, t_0)$ , or this subset is proper and almost all the algebras do not have this property. Moreover, if all the algebras of  $\mathfrak{N}_2(n, t)$  have the property  $\mathcal{S}(n_0, t_0)$ , then it is also true for any Lie algebra  $L \in \mathfrak{N}_2(n, t+k)$ , where  $k > 0$ , since for any  $k$ -dimensional subalgebra  $H < [L, L]$  we have  $L/H \in \mathfrak{N}_2(n, t)$ . Thus, we get the following.

**Proposition 3.** *If a ground field is algebraically closed, then for any positive integers  $n$ ,  $n_0$  and  $t_0$ , where  $n \geq n_0$  and  $t_0 \leq \frac{n_0(n_0-1)}{2}$ , there exists an integer  $C(n, n_0, t_0)$  such that for any integer  $t$  satisfying the inequality*

$$1 \leq t < C(n, n_0, t_0),$$

*a generic Lie algebra of  $\mathfrak{N}_2(n, t)$  doesn't have the property  $\mathcal{S}(n_0, t_0)$ , and the property  $\mathcal{S}(n_0, t_0)$  is true on  $\mathfrak{N}_2(n, t)$  if*

$$C(n, n_0, t_0) \leq t \leq \frac{n(n-1)}{2}.$$

The relations (1.4) and (1.5) give upper and lower bounds for  $C(n, n_0, t_0)$ . Computing  $C(n, n_0, t_0)$  precisely is subject to further investigation.

## 1.2 Finitely generated torsion-free nilpotent groups of class 2

Now let us introduce the notion of a generic property for nilpotent groups in an analogous way. Let  $G$  be a finitely generated 2-step nilpotent group without torsion. Denote by  $S = I(G') = \{x \in G \mid x^k \in G' \text{ for some } k\}$  the isolator of the commutator subgroup of  $G$ . Since the center of a torsion-free nilpotent group coincides with its isolator (see, for example, [1, Section 8]),  $I(G')$  also lies in the center. Hence  $S$  and the quotient group  $G/S$  are free abelian. Let  $b_1, \dots, b_t$  and  $a_1, \dots, a_n$  be bases of  $S$  and  $G$  modulo  $S$  respectively. Then the elements of  $G$  have the form

$$a_1^{k_1} \dots a_n^{k_n} b_1^{l_1} \dots b_t^{l_t}, \text{ with } k_i, l_j \in \mathbb{Z}, \text{ for } i = 1, \dots, n, j = 1, \dots, t. \quad (1.7)$$

Since the commutator map on a nilpotent group of class 2 is bilinear, we get

$$\begin{aligned} [x, y] &= [a_1^{k_1} \dots a_n^{k_n} b_1^{l_1} \dots b_t^{l_t}, a_1^{p_1} \dots a_n^{p_n} b_1^{q_1} \dots b_t^{q_t}] = \\ &= \prod_{i,j=1}^n [a_i, a_j]^{k_i p_j} = \prod_{l=1}^t b_l^{\sum_{i,j=1}^n k_i p_j \varphi_l(a_i, a_j)} = \prod_{l=1}^t b_l^{\varphi_l(xS, yS)}. \end{aligned} \quad (1.8)$$

for some  $t$ -tuple of integer skew-symmetric bilinear forms  $\Phi(G) = \{\varphi_1, \dots, \varphi_t\}$  on  $G/S$ .

Conversely, let  $\Phi = \{\varphi_1, \dots, \varphi_t\}$  be a  $t$ -tuple of integer skew-symmetric bilinear forms on a free abelian group  $\mathbb{Z}^n$  with a basis  $a_1, \dots, a_n$ . Consider the set  $G(\Phi)$  of formal products of the form (1.7). Define the product of two elements of  $G(\Phi)$  by

$$\begin{aligned} &a_1^{k_1} \dots a_n^{k_n} b_1^{l_1} \dots b_t^{l_t} \cdot a_1^{p_1} \dots a_n^{p_n} b_1^{q_1} \dots b_t^{q_t} = \\ &= a_1^{k_1+p_1} \dots a_n^{k_n+p_n} \prod_{k=1}^t b_k^{l_k+q_k+\sum_{i>j} k_i p_j \varphi_k(a_i, a_j)}. \end{aligned} \quad (1.9)$$

The product is well defined and it can be verified easily that the group axioms are satisfied. From (1.9) it follows that  $G(\Phi)$  is a nilpotent torsion-free group of class 2, that the subgroup  $S$  generated by  $b_1, \dots, b_t$  is central and freely generated by  $b_1, \dots, b_t$ ,

and the factor group  $G(\Phi)/S$  is abelian and freely generated by  $a_1S, \dots, a_nS$ . So we can naturally identify it with  $\mathbb{Z}^n$ . Using (1.9), we get

$$[a_i, a_j] = \prod_{l=1}^t b_l^{\varphi_l(a_i, a_j)} \quad \text{for } i, j = 1, \dots, n.$$

Further, since the commutator map is bilinear, we get that (1.8) holds.

It is easy to see that the  $t$ -tuple of bilinear forms corresponding to a new basis of  $S$  is defined by (1.3), where the transformation matrix  $C$  can be an arbitrary integer matrix with determinant 1. Therefore, as in the case of Lie algebras, we get the following property of  $\Phi(G)$ .

**Proposition 1'.** *The alternating bilinear forms of the  $t$ -tuple  $\Phi(G)$  corresponding to a group  $G$  are linearly independent. And conversely, if the forms of a  $t$ -tuple  $\Phi$  are linearly independent and  $G(\Phi)$  is a corresponding 2-step nilpotent group, then  $S = I((G(\Phi)))$ .*

The proof is carried over from Proposition 1.

Let  $\mathcal{N}_2(n, t)$  be the set of all 2-step nilpotent torsion free groups  $G$  with  $\text{rk } G' = t$  and  $\text{rk } G/I(G') = n$ . Denote by  $B_n(\mathbb{Z})$  the set of all integer bilinear forms on  $\mathbb{Z}^n$ . Put  $P_{n,t} = \{(\varphi_1, \dots, \varphi_t) \in B_n^t(\mathbb{Z}) \mid \varphi_1, \dots, \varphi_t \text{ are linearly independent}\}$ . Given a basis  $a_1, \dots, a_n$  of a vector space  $V$  over  $\mathbb{Q}$ , we can identify the elements of  $B_n(\mathbb{Z})$  with elements of  $B_n(\mathbb{Q})$  represented in this basis by integer matrices.

**Definition 1'.** We say that a property  $\mathcal{A}$  of  $\mathcal{N}_2(n, t)$  is true generically if the set of points of  $P_{n,t}$  corresponding to the groups without property  $\mathcal{A}$  is contained in some proper Zariski-closed subset of  $B_n^t(\mathbb{Q})$ .

A nilpotent group  $G \in \mathcal{N}_2(n, t)$  and the Lie algebra  $L = L(\Phi(G)) \in \mathfrak{N}_2^{\mathbb{Q}}(n, t)$ , defined by the same tuple of the forms, have quite similar properties. In particular,  $G$  contains an abelian subgroup of rank  $s$  if and only if  $L$  has an  $s$ -dimensional commutative subalgebra.  $G$  admits a surjective homomorphism onto a group of  $\mathcal{N}_2(n_0, t_0)$  if and only if  $L$  has the property  $\mathcal{S}(n_0, t_0)$ . Also we have  $\text{rk } Z(G) = \dim Z(L)$  and  $\text{rk } G' = \dim [L, L]$ . That is why the analogues of Theorems 1-3 holds true for the nilpotent groups. They do not need to be proved separately. This can be explained by the following arguments.

Denote by  $\sqrt{G}$  the Malcev completion of  $G \in \mathcal{N}_2(n, t)$ , that is the smallest complete nilpotent torsion free group containing  $G$  (see, for example, [1, Section 8]). Notice that  $\sqrt{G}$  can be considered as the group of elements of the form (1.7) with multiplication (1.9), where  $(\varphi_1, \dots, \varphi_t) = \Phi(G)$  and  $k_i, l_j, p_r, q_s \in \mathbb{Q}$ .

On the other hand, it is well known (see [1]) that in every nilpotent Lie  $\mathbb{Q}$ -algebra  $L$  one can define multiplication "o" by Campbell-Hausdorff formula in such a way that  $L$  is a torsion-free complete group, say  $L^\circ$ , of the same nilpotency class with respect to "o". (In the case of a 2-step nilpotent Lie algebra Campbell-Hausdorff formula has the

form  $x \circ y = x + y + \frac{1}{2}[x, y]$ .) The functor  $\Gamma : L \rightarrow L^\circ, (f : L_1 \rightarrow L_2) \rightarrow (f : L_1^\circ \rightarrow L_2^\circ)$  provides an isomorphism from category of the nilpotent class  $s$  Lie  $\mathbb{Q}$  algebras to the category of the nilpotent class  $s$  torsion-free complete groups. It is easy to see that  $\sqrt{G} \simeq L(\Phi(G))^\circ$ .

**Theorem 1'.** *For a generic group  $G \in \mathcal{N}_2(n, t)$  we have  $\text{rk } Z(G) = 2$  if  $t = 1$  and  $n$  is odd, and  $Z(G) = I(G')$  otherwise.*

**Theorem 2'.** *A generic group of  $\mathcal{N}_2(n, t)$  doesn't contain any abelian subgroup of rank  $\left\lfloor \frac{2n+t^2+3t}{t+2} \right\rfloor + 1$ .*

Also this theorem follows from [5].

**Theorem 3'.** *Suppose that the positive integers  $n, n_0, t$ , and  $t_0$  satisfy the inequality*

$$t < \frac{n(n-1)}{2} - \frac{nn_0}{t_0} + \frac{n_0^2}{t_0} + t_0 - \frac{n_0(n_0-1)}{2}.$$

*Then a generic group of  $\mathcal{N}_2(n, t)$  does not admit a surjective homomorphism onto a group of  $\mathcal{N}_2(n_0, t_0)$ .*

*If  $n, n_0, t$  and  $t_0$ , where  $n \geq n_0$ , satisfy the inequality*

$$t \geq \frac{n(n-1)}{2} - \frac{n_0(n_0-1)}{2} + t_0,$$

*then every group of  $\mathcal{N}_2(n, t)$  admits a surjective homomorphism onto a group of  $\mathcal{N}_2(n_0, t_0)$ .*

**Corollary.** *A generic group of  $\mathcal{N}_2(n, t)$  does not admit a surjective homomorphism on a non-abelian group of polycyclic rank  $N < n$ , if*

$$t < \frac{n^2}{2} - n \left( N - \frac{1}{2} \right) + \frac{N(N-1)}{2} + 1.$$

## 2 Preliminaries on the algebraic varieties

We recall that  $X \subset \mathbb{P}^n$  is a *closed in Zariski topology subset* if it consists of all points at which a finite number of homogeneous polynomials with coefficients in  $K$  vanishes. This topology induces Zariski topology on any subset of the projective space  $\mathbb{P}^n$ . A closed subset of  $\mathbb{P}^n$  is a *projective variety*, and an open subset of a projective variety is a *quasiprojective variety*. A nonempty set  $X$  is called *irreducible* if it cannot be written as the union of two proper closed subsets.

Further, let  $f : X \rightarrow \mathbb{P}^m$  be a map of a quasiprojective variety  $X \subset \mathbb{P}^n$  to a projective space  $\mathbb{P}^m$ . This map is *regular* if for every point  $x_0 \in X$  there exists a neighbourhood  $U \ni x_0$  such that the map  $f : U \rightarrow \mathbb{P}^m$  is given by an  $(m+1)$ -tuple

$(F_0 : \dots : F_m)$  of homogeneous polynomials of the same degree in the homogeneous coordinates of  $x \in \mathbb{P}^n$ , and  $F_i(x_0) \neq 0$  for at least one  $i$ .

We will use the following properties of the *dimension* of a quasiprojective variety.

- The dimension of  $\mathbb{P}^n$  is equal to  $n$ .
- If  $X$  is an irreducible variety and  $U \subset X$  is open, then  $\dim U = \dim X$ .
- The dimension of a reducible variety is the maximum of the dimension of its irreducible components.

For more details see [7]. We will need the following propositions which are also to be found in [7].

**Proposition 4.** *If  $X = \cup U_\alpha$  with open sets  $U_\alpha$ , and  $Y \cap U_\alpha$  is closed in  $U_\alpha$  for each  $U_\alpha$ , then  $Y$  is closed in  $X$ .*

**Proposition 5.** (see [4, Theorem 11.12]) Let  $f : X \rightarrow \mathbb{P}^n$  be a regular map of a quasiprojective variety  $X$ , and let  $Y$  be the closure of  $f(X)$ . For each point  $p \in X$ , we denote by  $X_p = f^{-1}(f(p)) \subset X$  the fiber of  $f$  containing  $p$ , and by  $\dim_p X_p$  denote the local dimension of  $X_p$  at the point  $p$ , that is, the maximal dimensions of an irreducible component of  $X_p$ , containing  $p$ . Then the set of all points  $p \in X$  such that  $\dim_p X_p \geq m$  is closed in  $X$  for any  $m$ . More over, if  $X_0$  is an irreducible component of  $X$  and  $Y_0 \subset Y$  is the closure of the image  $f(X_0)$ , then

$$\dim X_0 = \dim Y_0 + \min_{p \in X_0} \dim_p X_p. \quad (2.1)$$

**Proposition 6.** *Let  $f : X \rightarrow Y$  be a regular map between projective varieties, with  $f(X) = Y$ . Suppose that  $Y$  is irreducible, and that all the fibers  $f^{-1}(y)$  for  $y \in Y$  are irreducible and of the same dimension. Then  $X$  is irreducible.*

The following examples of projective varieties are important for our goals: a *direct product* of projective spaces, a *Grassmannian* and a *Shubert cell*.

Let  $\mathbb{P}^n, \mathbb{P}^m$  be projective spaces having homogeneous coordinates  $(u_0 : \dots : u_n)$  and  $(v_0 : \dots : v_m)$  respectively. Then the set  $\mathbb{P}^n \times \mathbb{P}^m$  of pairs  $(x, y)$  with  $x \in \mathbb{P}^n$  and  $y \in \mathbb{P}^m$  is naturally embedded as a closed set into the projective space  $\mathbb{P}^{(n+1)(m+1)-1}$  with homogeneous coordinates  $w_{ij}$  by the rule  $w_{ij}(u_0 : \dots : u_n; v_0 : \dots : v_m) = u_i v_j$ . Thus there is a topology on  $\mathbb{P}^n \times \mathbb{P}^m$ , induced by the Zariski topology on  $\mathbb{P}^{(n+1)(m+1)-1}$ .

**Proposition 7.** *A subset  $X \subset \mathbb{P}^n \times \mathbb{P}^m \subset \mathbb{P}^N$  is a closed algebraic subvariety if and only if it is given by a system of equations*

$$G_i(u_0 : \dots : u_n; v_0 : \dots : v_m) = 0 \quad (i = 1, \dots, t)$$

*homogeneous in each set of variables  $u_i$  and  $v_j$ .*

**Proposition 8.** Consider an  $n$ -dimensional vector space  $V$  with a basis  $\{e_1, \dots, e_n\}$ . Let  $U$  be a  $k$ -dimensional subspace of  $V$  with a basis  $\{f_1, \dots, f_k\}$ . To  $U$  we assign the point  $P(U)$  of the projective space  $\mathbb{P}(\Lambda^k V)$  by the rule

$$P(U) = f_1 \wedge \dots \wedge f_k.$$

The point  $P(U)$  has the following form in the basis  $\{e_{i_1} \wedge \dots \wedge e_{i_k}\}_{i_1 < \dots < i_k}$  of  $\Lambda^k V$ :

$$P(U) = \sum_{i_1 < \dots < i_k} p_{i_1 \dots i_k} e_{i_1} \wedge \dots \wedge e_{i_k}.$$

Then the homogeneous coordinates  $p_{i_1 \dots i_k}$  of  $P(U)$  are called the Plucker coordinates of  $U$ ,  $P(U)$  is uniquely determined by  $U$ , and the following assertions hold.

- (i) The subset of all points  $p \in \mathbb{P}(\Lambda^k V)$  of the form  $p = P(U)$  is closed in  $\mathbb{P}(\Lambda^k V)$ ; this subset  $G(k, n)$  (the Grassmannian or Grassmann variety) is defined by the relations

$$\sum_{r=1}^{k+1} (-1)^r p_{i_1 \dots i_{k-1} j_r} p_{j_1 \dots \widehat{j_r} \dots j_{k+1}} = 0 \quad (2.2)$$

for all sequences  $i_1 \dots i_{k-1}$  and  $j_1 \dots j_{k+1}$ .

- (ii)  $\dim G(k, n) = k(n - k)$ .

- (iii)  $G(k, n)$  is irreducible (see Section 2.2.7 of [8]).

- (iv) Suppose, for example, that  $p_{1 \dots k} \neq 0$ . If  $p = (p_{i_1 \dots i_r}) = P(U)$ , then  $U$  has a basis  $\{f_1, \dots, f_k\}$  such that

$$f_i = e_i + \sum_{r > k} a_{ir} e_r \quad \text{for } i = 1, \dots, k, \quad (2.3)$$

where

$$a_{ir} = (-1)^{k-i} \frac{p_{1 \dots \widehat{i} \dots k r}}{p_{1 \dots k}}. \quad (2.4)$$

(Here  $\widehat{i}$  means that the index  $i$  is discarded.)

**Proposition 9.** (see [2, Section 14.7] or [3, Section 1.5]) Consider an  $n$ -dimensional vector space  $V$ . For any increasing sequence

$$0 \subsetneq V_1 \subsetneq \dots \subsetneq V_k$$

of subspaces of  $V$  put

$$W(V_1, \dots, V_k) = \{p = P(U) \in G(k, n) \mid \dim(U \cap V_i) \geq i \text{ for } i = 1, \dots, k\}.$$

Then all the subsets  $W(V_1, \dots, V_k)$ , called Schubert cells, are closed in  $G(k, V)$ , and

$$\dim W(V_1, \dots, V_k) = \sum_{i=1}^k (\dim V_i - i). \quad (2.5)$$



**Corollary.** Let  $V$  be a vector space of dimension  $n$ , and let  $s_0 = \max\{0, k + m - n\}$ . Given an  $m$ -dimensional subspace  $U \subset V$  ( $m > 0$ ), put

$$G_s = G_s(U, V, k) = \{P(U') \in G(k, n) \mid \dim U \cap U' \geq s\} \quad (2.6)$$

for  $s = s_0, s_0 + 1, \dots, \min\{k, m\}$ . Then the sets  $G_s$  are closed in  $G(k, n)$ , and

$$\dim G_s = s(m - s) + (k - s)(n - k). \quad (2.7)$$

*Proof.* It is easy to check that the sets  $G_s$  are Schubert cells. Indeed, choose a basis  $\{e_1, \dots, e_n\}$  of  $V$  such that  $e_1, \dots, e_m$  span  $U$ . Then  $G_s = W(V_1, \dots, V_k)$  where

$$\begin{aligned} V_i &= \langle e_1, \dots, e_{m-s+i} \rangle & \text{for } i = 1, \dots, s, \\ V_i &= \langle e_1, \dots, e_{n-k+i} \rangle & \text{for } i = s+1, \dots, k. \end{aligned}$$

(The conditions  $\dim(U' \cap V_i) \geq i$  for  $i > s$  are trivial, and for  $i \leq s$  they follow from the condition  $\dim(U' \cap V_s) = \dim(U' \cap U) \geq s$ .) Using (2.5), we get

$$\dim G_s = \sum_{i=1}^s (m - s) + \sum_{i=s+1}^k (n - k) = s(m - s) + (k - s)(n - k). \quad \square$$

### 3 Proofs

**Lemma 1.** Let  $K$  be an infinite field. Denote by  $\bar{K}$  its algebraic closure. Suppose that a generic Lie algebra of  $\mathfrak{N}_2^K(n, t)$  has property  $\mathcal{A}$  and for any Lie algebra  $L \in \mathfrak{N}_2^K(n, t)$  without property  $\mathcal{A}$  the Lie algebra  $\bar{L} = L_K \otimes \bar{K} \in \mathfrak{N}_2^{\bar{K}}(n, t)$  does not have property  $\mathcal{A}$  too. Then  $\mathcal{A}$  is also a generic property of  $\mathfrak{N}_2^K(n, t)$ .

*Proof.* The  $t$ -tuples  $\Phi(L)$  and  $\Phi(\bar{L})$ , corresponding to the Lie algebras  $L \in \mathfrak{N}_2^K(n, t)$  and  $\bar{L} \in \mathfrak{N}_2^{\bar{K}}(n, t)$  respectively, can be defined by the same  $t$ -tuple of matrices with coefficients in  $K$ . So, under the condition of the Lemma, the set of points of  $M_{n,t}(K)$  corresponding to the algebras of  $\mathfrak{N}_2^K(n, t)$  without property  $\mathcal{A}$  satisfies some finite system of homogeneous equations, say  $(*)$ , over  $\bar{K}$ . If the variables take values in  $K$ ,  $(*)$  is equivalent to some finite system of homogeneous equations over  $K$ . The last system is not zero on  $M_{n,t}(K)$  identically, since  $K$  is infinite and  $(*)$  defines proper subset of  $M_{n,t}(\bar{K})$ .  $\square$

It follows from Lemma 1 that it is enough to prove Theorems 1-3 just for the case of algebraically closed ground field  $K$ . So, from now on if not stated otherwise we will assume that  $K$  is algebraically closed.

Propositions 1 and 2 show that we can define the correspondence between the elements of  $\mathfrak{N}_2(n, t)$  and  $G(t, B_n)$  by the rule  $\varphi(L) = \varphi_1 \wedge \dots \wedge \varphi_t$  for  $L \in \mathfrak{N}_2(n, t)$  if  $\Phi(L) = (\varphi_1, \dots, \varphi_t)$ , and vice versa,  $L(\varphi) = L(\varphi_1, \dots, \varphi_t)$  for  $\varphi = \varphi_1 \wedge \dots \wedge \varphi_t \in G(t, B_n)$ .

**Lemma 2.**  $\mathcal{A}$  is a generic property of  $\mathfrak{N}_2(n, t)$  if and only if the set of points of  $G(t, B_n)$  corresponding to the algebras without property  $\mathcal{A}$  belongs to some proper Zariski-closed subset of  $G(t, B_n)$ .

*Proof.* Consider a regular map  $f : M_{n,t} \rightarrow G(t, B_n)$  such that  $f$  takes each  $t$ -tuple of alternating bilinear forms to the subspace spanned by it, that is,

$$f(\varphi_1, \dots, \varphi_t) = \varphi_1 \wedge \dots \wedge \varphi_t.$$

By definition, Lie algebras  $L(\Phi)$  and  $L(f(\Phi))$  corresponding to the points  $\Phi \in M_{n,t}$  and  $f(\Phi)$  respectively are isomorphic.

Denote by  $M(\mathcal{A}) \subset M_{n,t}$  and  $G(\mathcal{A}) \subset G(t, B_n)$  the sets of points corresponding to the algebras without property  $\mathcal{A}$ . Clearly,  $f^{-1}(G(\mathcal{A})) = M(\mathcal{A})$ . Suppose that  $G(\mathcal{A})$  belongs to some proper closed subset  $X \subsetneq G(t, B_n)$ . Then  $M(\mathcal{A})$  is contained in  $f^{-1}(X)$ . Since  $f$  is surjective and continuous,  $f^{-1}(X)$  is a proper closed subset of  $M_{n,t}$ .

Inversely, suppose that  $M(\mathcal{A})$  belongs to some proper closed subset  $Y \subsetneq M_{n,t}$ . A fiber  $f^{-1}(\varphi)$  consists of all the bases of a given  $t$ -dimensional vector space considered up to proportionality. It can be parametrized by invertible matrices of size  $t \times t$ . Hence, all the fibers  $f^{-1}(\varphi)$  for  $\varphi \in G(t, B_n)$  are isomorphic and of the same dimension  $t^2 - 1$ .

Obviously,  $L(\Phi)$  does not have property  $\mathcal{A}$  only if  $f^{-1}(f(\Phi)) \subset Y$ . Hence, we can consider a set  $Y_0 = \{\Phi \in Y \mid \dim_{\Phi}(f^{-1}(f(\Phi)) \cap Y) = t^2 - 1\}$  instead of  $Y$ . An application of Proposition 5 to the map  $\bar{f} = f|_Y$  yields that  $Y_0$  is also closed.

Denote by  $X_0$  the closure of  $f(Y_0)$  in  $G(t, B_n)$ . We have  $G(\mathcal{A}) \subset X_0$ . Let us show that  $X_0$  is proper subset of  $G(t, B_n)$ . Indeed, by (??), for any irreducible component  $Y'_0 \subset Y_0$  we have

$$\dim Y'_0 = \dim X'_0 + t^2 - 1,$$

where  $X'_0 \subset X_0$  is a closure of the image  $f(Y'_0)$ . Recalling that  $\dim M_{n,t} = \dim G(t, B_n) + t^2 - 1$  and  $\dim Y'_0 < \dim M_{n,t}$ , we obtain  $\dim X'_0 < \dim G(t, B_n)$ . Consequently,  $\dim X_0 < \dim G(t, B_n)$ .  $\square$

### 3.1 Proof of Theorem 1

Consider a Lie algebra  $L \in \mathfrak{N}_2(n, t)$ . Let  $S = [L, L]$ ,  $L = V \oplus S$  and  $\Phi(L) = (\varphi_1, \dots, \varphi_t)$ . Fix a basis  $e_1, \dots, e_n$  for  $V$  and identify forms  $\varphi_i$  with their matrices with respect to this basis.  $Z(L)$  is strictly greater than  $S$  if and only if there exists nonzero element  $c \in V$  such that

$$\varphi_i(c, e_j) = 0 \quad i = 1, \dots, t, \quad j = 1, \dots, n. \quad (3.1)$$

Denote by  $D$  the subset of  $H = \mathbb{P}(V) \times \mathbb{P}(B_n^t)$  consisting of pairs  $(c, \varphi)$  satisfying (3.1). The system of equations (3.1) is linear in each set of variables  $c$  and  $\varphi$ . Therefore, by Proposition 7,  $D$  is a projective variety.

Consider the projection  $\pi : D \rightarrow \mathbb{P}(B_n^t)$ . Since  $\pi$  is a regular map,  $\pi(D)$  is closed in  $\mathbb{P}(B_n^t)$ . Moreover,  $\pi(D) \cap M_{n,t}$  consists of  $t$ -tuples  $(\varphi_1, \dots, \varphi_t)$  corresponding to the algebras with non-trivial center modulo the commutator ideal. Obviously,  $\pi(D) \cap M_{n,t}$  is a proper subset except the case when  $t = 1$  and  $n$  is odd in which the only form  $\varphi_1$

defining a Lie algebra is always degenerate. In the last case the dimension of the center of a Lie algebra is greater than 2 only if the rank of  $\varphi_1$  is strictly less than  $n - 1$ . This condition defines a proper closed subset in  $M_{n,1}$ .  $\square$

### 3.2 Proof of Theorem 2

*Proof of Theorem 2.* The Main Lemma of [6] states that if the positive integers  $t \geq 2$ ,  $k$  and  $n$  satisfy the inequality

$$2n \geq t(k - 1) + 2k, \quad (3.2)$$

then for any  $t$ -tuple  $\{\varphi_1, \dots, \varphi_t\} \in B_n^t(K)$  there exists a  $k$ -dimensional subspace that is simultaneously isotropic for all of the forms  $\varphi_1, \dots, \varphi_t$ , i.e. on which all the forms are zero.

It follows easily from the proof of the Main Lemma that the set  $\pi_2(S)$  (in the notation of Section 2.2 of [6]) consisting of all the  $t$ -tuples of forms with common  $k$ -dimensional isotropic subspace is a Zariski-closed subset of  $\mathbb{P}(B_n^t)$ . Furthermore, we have  $\pi_2(S) = \mathbb{P}(B_n^t)$  if  $t > 1$  and the inequality (3.2) holds, and  $\pi_2(S)$  is a proper subset if (3.2) is not true.

Let  $L(\Phi) = V \oplus S \in \mathfrak{N}_2(n, t)$  be a Lie algebra associated with a tuple  $\Phi$ . A subalgebra  $H \leq L$  is commutative if and only if the subspace  $H/(H \cap S)$  is isotropic for all of the forms of  $\Phi$ . Therefore each algebra of  $\mathfrak{N}_2(n, t)$  contains a commutative subalgebra of dimension  $s = k + t$ , where  $k$  is the maximal integer satisfying (3.2), that is  $k = \lfloor \frac{2n+t}{t+2} \rfloor$ . And a generic Lie algebra of  $\mathfrak{N}_2(n, t)$  has no abelian subalgebras of dimension greater than  $s$ .  $\square$

If the ground field  $K$  is not algebraically closed, then the maximal dimension of commutative subalgebras of a Lie algebra  $L$  over  $K$  may be strictly less than that of  $\bar{L} = L_K \otimes \bar{K}$  over  $\bar{K}$ . And so the condition on the ground field to be closed in the first part of Theorem 2 is essential.

*Example.* Let  $K$  be a subfield of the field of real numbers. There exists a Lie algebra  $L \in \mathfrak{N}_2^K(4, 3)$  without commutative subalgebras of dimension 5.

*Proof.* Consider a Lie algebra  $L = V \oplus S$  defined by the following 3-tuple of matrices

$$\varphi_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}, \quad \varphi_2 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \quad \varphi_3 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}.$$

Elements  $x, y \in V$  commute if and only if  $\varphi_i(x, y) = 0$  for  $i = 1, 2, 3$ . Hence the set of all elements  $y \in V$  commuting with a given element  $x$  with coordinates  $(x_1, x_2, x_3, x_4)$  is a solution space of the system of homogeneous linear equations with the matrix

$$M(x) = \begin{pmatrix} x_2 & -x_1 & x_4 & -x_3 \\ x_3 & -x_4 & -x_1 & x_2 \\ x_4 & x_3 & -x_2 & -x_1 \end{pmatrix}.$$

The system has a solution not proportional to  $x$  if and only if  $M(x)$  is rank deficient. Let us compute  $3 \times 3$  minors of  $M(x)$ :

$$A_i = (-1)^i x_i (x_1^2 + x_2^2 + x_3^2 + x_4^2) \quad \text{for } i = 1, 2, 3, 4.$$

Obviously, if all  $A_i = 0$  and  $x_i \in \mathbb{R}$ , then  $x_1 = x_2 = x_3 = x_4 = 0$ . Consequently,  $L$  has no commutative subalgebras of dimension greater than 1 modulo commutator ideal.  $\square$

### 3.3 Proof of Theorem 3

For any subspace  $U$  of an  $n$ -dimensional vector space  $V$ , put

$$N_0(U) = \{\varphi \in B_n \mid \varphi(x, u) = 0 \text{ for } x \in V \text{ and } u \in U\}.$$

Choose a basis  $e_1, \dots, e_n$  of  $V$  such that  $e_1, \dots, e_k$  span  $U$ . Now an alternating bilinear form  $\psi$  lies in  $N_0(U)$  if and only if the matrix of  $\psi$  is zero except lower right-hand  $(n - k) \times (n - k)$  submatrix in this basis. Thus,

$$\dim N_0(U) = \frac{(n - k)(n - k - 1)}{2}. \quad (3.3)$$

**Lemma 3.** *Consider a Lie algebra  $L \in \mathfrak{N}_2(n, t)$ . Let  $S = [L, L]$ ,  $L = V \oplus S$  and  $\Phi(L) = (\varphi_1, \dots, \varphi_t)$ .  $L$  has a property  $\mathcal{S}(n_0, t_0)$  if and only if there exists a subspace  $U \subset V$  of dimension  $n - n_0$  such that*

$$\dim(N_0(U) \cap \langle \varphi_1, \dots, \varphi_t \rangle) \geq t_0. \quad (3.4)$$

*Proof.*  $L$  has property  $\mathcal{S}(n_0, t_0)$  if and only if there exists an ideal  $I \triangleleft L$  of dimension  $(n + t - n_0 - t_0)$  such that  $\dim(I \cap S) = t - t_0$ . Indeed, in this case we have  $L/I \in \mathfrak{N}_2(n_0, t_0)$ , since the image of a commutator ideal is the commutator ideal of the image. And so we can take  $I$  as a kernel of a desired surjective map.

Assume that such  $I$  exists. Choose a basis  $z'_1, \dots, z'_t$  of  $S$  such that  $z'_{t_0+1}, \dots, z'_t \in (S \cap I)$ . Let  $(\varphi'_1, \dots, \varphi'_t)$  be a  $t$ -tuple of forms associated with  $L$  in this basis. Denote by  $U$  the projection of  $I$  on  $V$  along  $S$ . Clearly,  $\dim U = n - n_0$ . Since for any  $x \in V$  and  $u \in U$  we have  $[x, u] = \sum_{i=1}^t \varphi'_i(x, u) z'_i \in \langle z'_{t_0+1}, \dots, z'_t \rangle$ , we get  $\varphi'_1, \dots, \varphi'_{t_0} \in N_0(U)$ . It follows from (1.3) that  $\varphi'_1, \dots, \varphi'_{t_0}$  are linear combinations of  $\varphi_1, \dots, \varphi_t$  and, by Proposition 1, they are linearly independent. Consequently, (3.4) holds.

On the contrary, let  $U$  be as in the lemma. Choose  $t_0$  linearly independent forms  $\varphi'_1, \dots, \varphi'_{t_0} \in (N_0(U) \cap \langle \varphi_1, \dots, \varphi_t \rangle)$  and extend them to a basis  $\varphi'_1, \dots, \varphi'_t$  of  $\langle \varphi_1, \dots, \varphi_t \rangle$ . It follows from (1.2) and (1.3) that we can choose a basis  $z'_1, \dots, z'_t$  of  $S$  such that  $\Phi(L) = (\varphi'_1, \dots, \varphi'_t)$  in this basis. For any  $x \in V$  and  $u \in U$  we have  $[x, u] = \sum_{i=t_0+1}^t \varphi'_i(x, u) z'_i \in \langle z'_{t_0+1}, \dots, z'_t \rangle$ . Hence, the linear span  $I = \langle U, z'_{t_0+1}, \dots, z'_t \rangle$  is a desired ideal.  $\square$

**Lemma 4.** *Suppose that the positive integers  $n$ ,  $n_0$ ,  $t$  and  $t_0$  satisfy the inequality (1.5) and let  $n \geq n_0$ . Then any Lie algebra  $L \in \mathfrak{N}_2(n, t)$  has the property  $\mathcal{S}(n_0, t_0)$ .*

*Proof.* In the notation of Lemma 3, for any Lie algebra  $L \in \mathfrak{N}_2(n, t)$  and for any subspace  $U \subset V$  of dimension  $n - n_0$  we have

$$\begin{aligned} \dim(N_0(U) \cap \langle \varphi_1, \dots, \varphi_t \rangle) &\geq \dim N_0(U) + \dim \langle \varphi_1, \dots, \varphi_t \rangle - \dim B_n = \\ &\stackrel{(3.3)}{=} \frac{n_0(n_0 - 1)}{2} + t - \frac{n(n - 1)}{2}. \end{aligned}$$

Combining this with inequality (1.5), we get (3.4).  $\square$

Obviously, if  $n < n_0$  or  $t < t_0$ , there are no Lie algebras in  $\mathfrak{N}_2(n, t)$  having the property  $\mathcal{S}(n_0, t_0)$ . And so in what follows to prove the theorem we fix integers  $n, n_0, t, t_0, k$  such that  $n = n_0 + k, t \geq t_0 \geq 1, n_0 \geq 2, k \geq 0$  and an  $n$ -dimensional vector space  $V$ . Denote by  $D$  the following subset of the direct product  $H = G(k, V) \times G(t, B_n) \subset \mathbb{P}(\Lambda^k V) \times \mathbb{P}(\Lambda^t B_n)$

$$D = \{(p, \varphi) \in H \mid p = P(U), \varphi = P(\Omega), \dim(N_0(U) \cap \Omega) \geq t_0\}.$$

**Lemma 5.**  *$D$  is a projective variety.*

*Proof.* Fix a basis  $e_1, \dots, e_n$  of  $V$  and a basis  $\psi_1, \dots, \psi_{\frac{n(n-1)}{2}}$  of  $B_n$ . Then the coordinates  $(p_{i_1 \dots i_k})$  and  $(\varphi_{j_1 \dots j_t})$  on  $\mathbb{P}(\Lambda^k V)$  and  $\mathbb{P}(\Lambda^t B_n)$  respectively (and, hence, on  $H$ ) are naturally defined.

Let us prove that  $D$  is closed in  $H$ . For this we consider the covering of the projective variety  $H$  by the open sets

$$O_{i_1 \dots i_k, j_1 \dots j_t} = \{(p, \varphi) \in H \mid p_{i_1 \dots i_k} \neq 0 \text{ and } \varphi_{j_1 \dots j_t} \neq 0\}$$

and verify that  $O_{i_1 \dots i_k, j_1 \dots j_t} \cap D$  is closed in  $O_{i_1 \dots i_k, j_1 \dots j_t}$  for any sequences  $i_1 \dots i_k$  and  $j_1 \dots j_t$  with  $i_1 < \dots < i_k$ , and  $j_1 < \dots < j_t$ . Then, by Proposition 4,  $D$  is closed in  $H$ . Hence,  $D$  is also a projective variety.

Let us consider an arbitrary point  $(p, \varphi) \in G(k, V) \times G(t, B_n)$ . We can assume without loss of generality that  $p_{1 \dots k} \neq 0$  and  $\varphi_{1 \dots t} \neq 0$ . Let  $p = P(U)$  and  $\varphi = P(\Omega)$ . Then, according formula (2.3),  $U$  has the basis

$$f_i = e_i + \sum_{r=k+1}^n a_{ir} e_r \quad \text{for } i = 1, \dots, k, \quad (3.5)$$

where

$$a_{ir} = (-1)^{k-i} \frac{p_{1 \dots \hat{i} \dots kr}}{p_{1 \dots k}}; \quad (3.6)$$

and  $\Omega$  has the basis

$$\varphi_i = \psi_i + \sum_{r=t+1}^{\frac{n(n-1)}{2}} b_{ir} \psi_r \quad \text{for } i = 1, \dots, t, \quad (3.7)$$

where

$$b_{ir} = (-1)^{t-i} \frac{\varphi_{1 \dots \hat{i} \dots tr}}{\varphi_{1 \dots t}}. \quad (3.8)$$

It follows from (3.5) that the vectors  $f_1, \dots, f_k, e_{k+1}, \dots, e_n$  form a basis of  $V$ . Denote by  $C$  the transformation matrix from the basis  $e_1, \dots, e_n$  to this new one and by  $A_1, \dots, A_{\frac{n(n-1)}{2}}$  the matrices of  $\psi_1, \dots, \psi_{\frac{n(n-1)}{2}}$  in the initial basis. Then, by formula (3.7), matrices of  $\varphi_1, \dots, \varphi_t$  have the form

$$F_i = C^\perp \left( A_i + \sum_{r=t+1}^{\frac{n(n-1)}{2}} b_{ir} A_r \right) C \quad \text{for } i = 1, \dots, t,$$

in the new basis and, as we mentioned above,  $N_0(U)$  consists of the forms whose matrices are zero except lower right-hand  $(n-k) \times (n-k)$  submatrix.

Let  $E_1, \dots, E_{\frac{n_0(n_0-1)}{2}}$  be a basis of  $N_0(U)$ . (These are matrices with constant coefficients.) The condition  $\dim(N_0(U) \cap \Omega) \geq t_0$  holds if and only if the rank of the vector system  $\{F_1, \dots, F_t, E_1, \dots, E_{\frac{n_0(n_0-1)}{2}}\}$  is not greater than  $s = \frac{n_0(n_0-1)}{2} + t - t_0$ . The last condition in turn is equivalent to the fact that all the minors of size  $s+1$  in the corresponding matrix are zero. That is, we have a system of polynomial equations in variables  $a_{ir}$  and  $b_{ir}$ . Replacing this variables using (3.6) and (3.8) and multiplying both sides of the equations by the appropriate powers of  $p_{1\dots k}$  and  $\varphi_{1\dots t}$ , we obtain the system of equations homogeneous separately in each set of variables  $p$  and  $\varphi$ . According to Proposition 7, the set  $D \cap O_{1\dots k, 1\dots t}$ , defined by the last system, is closed in  $O_{1\dots k, 1\dots t}$ .  $\square$

Now consider the projections  $\pi_1 : D \rightarrow \mathbb{P}(\Lambda^k V)$  and  $\pi_2 : D \rightarrow \mathbb{P}(\Lambda^t B_n)$  such that  $\pi_1(p, \varphi) = p$ ,  $\pi_2(p, \varphi) = \varphi$ . These are regular maps. Clearly  $\pi_1(D) = G(k, V)$ .

**Lemma 6.**  $\varphi_1 \wedge \dots \wedge \varphi_t \in G(t, B_n)$  belongs to  $\pi_2(D)$  if and only if the corresponding Lie algebra  $L(\varphi_1, \dots, \varphi_t)$  has the property  $\mathcal{S}(n_0, t_0)$ .  $\pi_2(D)$  is closed in  $G(t, B_n)$ .

*Proof.* The lemma follows from Lemma 3 and from the fact that the image of the projectiv variety  $D$  under the regular map  $\pi_2$  is closed.  $\square$

**Lemma 7.** If the integers  $n$ ,  $n_0$ ,  $t$  and  $t_0$  do not satisfy the relation (1.5), then for any point  $p \in G(k, V)$  the fiber  $\pi_1^{-1}(p)$  is an irreducible projective variety of dimension

$$\dim \pi_1^{-1}(p) = t_0 \left( \frac{n_0(n_0-1)}{2} - t_0 \right) + (t - t_0) \left( \frac{n(n-1)}{2} - t \right). \quad (3.9)$$

*Proof.* For any point  $p = P(U) \in G(k, V)$  we have

$$\pi_1^{-1}(p) = \{P(\Omega) \in G(t, B_n) \mid \dim(N_0(U) \cap \Omega) \geq t_0\}.$$

That is, the fiber  $\pi_1^{-1}(p)$  is a set of type (2.6):

$$\pi_1^{-1}(p) = G_{t_0}(N_0(U), B_n, t).$$

Since (1.5) doesn't holds,  $t_0$  satisfies the condition of the Corollary of Proposition 9. We apply this corollary, taking in account that, by (3.3),  $\dim N_0(U) = \frac{n_0(n_0-1)}{2}$  and  $\dim B_n = \frac{n(n-1)}{2}$ , and thus obtain (3.9).  $\square$

**Lemma 8.**  *$D$  is an irreducible variety of dimension*

$$\dim D = t_0 \left( \frac{n_0(n_0 - 1)}{2} - t_0 \right) + (t - t_0) \left( \frac{n(n - 1)}{2} - t \right) + n_0(n - n_0). \quad (3.10)$$

*Proof.* By the previous lemma, all the fibers  $\pi_1^{-1}(p)$  are irreducible and of the same dimension. The image  $\pi_1(D) = G(k, V)$  is also irreducible by Proposition 8 (iii). So, by Proposition 6,  $D$  is irreducible and we can use formula (??) to compute its dimension. Namely,  $\dim D = \dim \pi_1^{-1}(p) + \dim G(k, V)$ . Using Lemma 7 and Proposition 8 (ii), we get (3.10).  $\square$

*Proof of Theorem 3.* To prove the first part of the theorem it is enough to show that under the condition (1.4) the inequality  $\dim \pi_2(D) < \dim G(t, B_n)$  holds and apply Lemmas 2 and 6. By (??), this inequality follows from

$$\dim D < \dim G(t, B_n).$$

Substituting for  $\dim D$  using Lemma 8 and recalling that  $\dim G(t, B_n) = t \left( \frac{n(n-1)}{2} - t \right)$  we obtain the relation equivalent to (1.4).

The second part of the theorem follows from Lemma 4. So, Theorem 3 is proved.  $\square$

*Proof of the Corollary of Theorem 3.* A Lie algebra  $L$  does not admit a surjective homomorphism on a non-abelian Lie algebra of dimension  $N$ , if it does not have the property  $\mathcal{S}(n_0, t_0)$  for any positive integers  $n_0, t_0$  such that  $n_0 + t_0 = N$  and  $t_0 \leq \frac{n_0(n_0-1)}{2}$ . By Theorem 3, this condition holds for almost all the Lie algebras  $L \in \mathfrak{N}_2(n, t)$  if for all of the stated values of  $n_0$  and  $t_0$  we have (1.4), that is,

$$t < \frac{n^2}{2} + \frac{1}{2}n - 2n_0 - \frac{N(n - N)}{N - n_0} - \frac{n_0(n_0 - 1)}{2}.$$

If  $N \leq n$ , the right term of the inequality is decreasing in  $n_0$  function and, hence, it takes the minimum value at  $n_0 = N - 1$ .  $\square$

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